Particle hopping models and traffic flow theory

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This paper shows how particle hopping models fit into the context of traffic flow theory, that is, it shows connections between fluid-dynamical traffic flow models, which derive from the Navier-Stokes equations, and particle hopping models. In some cases, these connections are exact and have long been established, but have never been viewed in the context of traffic theory. In other cases, critical behavior of traffic jam clusters can be compared to instabilities in the partial differential equations. Finally, it is shown how all this leads to a consistent picture of traffic jam dynamics. In consequence, this paper starts building a foundation of a comprehensive *dynamic* traffic theory, where strengths and weaknesses of different models (fluid-dynamical, car-following, particle hopping) can be compared, and thus allowing to *systematically* chose the appropriate model for a given question. [S1063-651X(96)06805-5]

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I. INTRODUCTION

Traffic jams have always been annoying. At least in the industrialized countries, the standard reaction has been to expand the transportation infrastructure to match demand. In this phase of fast growth, relatively rough planning tools were sufficient. However, in recent years most industrialized societies started to see the limits of such growth. In densely populated areas, there is only limited space available for extensions of the transportation system and we face increasing pollution and growing accident frequencies as the downsides of mobility. In consequence, planning is now turning to a fine tuning of the existing systems, without major extensions of facilities. This is, for example, reflected in the United States by the Clean Air Act and by the Intermodal Surface Transportation and Efficiency Act legislation. The former sets standards of air quality for urban areas, whereas the latter forces planning authorities to evaluate land use policies, intermodal connectivity, and enhanced transit service when planning transportation.

In consequence, planning and prediction tools with a much higher reliability than in the past are necessary. Due to the high complexity of the problems, analytical approaches are infeasible. Current approaches are simulation based (e.g., [1–4]), which is driven by necessity, but largely enhanced by the widespread availability of computing power nowadays. Yet, also for computers one needs good simplified models of the phenomena of interest: Just coding a perfect representation of reality into the computer is not possible because of limits of knowledge, limits of human resources for coding all these details, and limits of computational resources.

Practical simulation has to observe tradeoffs between *resolution*, *fidelity*, and *scale* [5]. Resolution refers to the smallest entities (objects, particles, and processes) resolved in a simulation, whereas fidelity means the degree of realism in modeling each of these entities, and scale means the (spatial, temporal, etc.) size of the problem. It is empirically well known, for example, from fluid dynamics, that to a certain extent a low-fidelity high-resolution model (lattice gas automata [6,7]) can do as well as a high-fidelity low-resolution model (discretization of the Navier-Stokes equations) or, in short, resolution can replace fidelity.

Current state-of-the-art traffic modeling has a fixed unit of (minimal) resolution and that is the individual traveler. Since one is aiming for rather large scales (for example, the Los Angeles area consists of approximately 10 million potential travelers), it is rather obvious that one has to sacrifice fidelity to achieve reasonable computing times.

One important part of transportation modeling is road traffic. For example, in Germany, road traffic currently contributes more than 81% of all passenger and 52.7% of all freight transportation [8]. Despite widespread efforts, the share of road transportation is still increasing. For that reason, it makes sense to start with road traffic when dealing with transportation systems.

Putting these arguments together, one thing that is needed for large-scale transportation simulations is a *minimal* representation of road traffic. Particle hopping models clearly are candidates for this, and even if not, building a *minimal* theory of road traffic is certainly the right starting point.

This paper shows how particle hopping models fit into the context of traffic flow theory. It starts out with a historical overview of traffic flow theory (Sec. II), followed by a systematic review of fluid-dynamical models for traffic flow (Sec. III) starting from the Navier-Stokes equations. Section IV defines different particle hopping models that are of interest in the context of traffic flow. Section V then shows the different connections between the fluid-dynamical traffic flow models and particle hopping models. In some cases, these connections are exact and have long been established, but have never been viewed in the context of traffic theory. In other cases, critical behavior of traffic jam clusters can be compared to instabilities in the partial differential equations. Finally, it is shown how this leads to a consistent picture of traffic jam dynamics (Sec. VI). A discussion of the consequences for traffic simulations (Sec. VII) serves as summary

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and discussion, and a collection of open questions (Sec. VIII) conclude the paper.

II. HISTORICAL OVERVIEW OF TRAFFIC THEORY

Vehicular traffic has been a widely and thoroughly researched area in the 1950s and 1960s. For a review of traffic theory, see, for example, one of [9–11]. Vehicular traffic theory can be broadly separated into three branches: traffic *flow* theory, *car-following* theory, and one more recent addition, particle hopping models.

A. Traffic *flow* theory

Traffic flow theory is concerned with finding relations between the three fundamental variables of traffic flow, which are velocity v, density ρ, and current or throughput or flow j. Only two of these variables are independent since they are related through $j=\rho v$. Possible units for these variables are $[v]=$ km/h, $[\rho]=$vehicles/km, and $[j]=$ vehicles/h.

The first approach of traffic flow theory historically was to search for *time-independent* relations between j, ρ, and v. These relations are the so-called fundamental diagrams. The form of such a relation is, though, still debated in the traffic flow literature [12,13]. The problem stems mainly from the fact that reality measurements are done in nonstationary conditions. There, only short time averages make sense and they usually show large fluctuations. I will, at the end of the paper, discuss how a dynamic, particle-based description of traffic can resolve these difficulties.

The second step of traffic flow theory was to introduce a dynamic, i.e., time-dependent description. This was achieved by a well-known paper from Lighthill and Whitham, published in 1955 [14]. This paper introduced a description based on the equation of continuity, together with the assumption that flow (or velocity) depends on the density only, i.e., there is no relaxation time, velocity adapts *instantaneously* to the surrounding density.

Prigogine and Herman developed a kinetic theory for traffic flow [15]. They derived the Lighthill-Whitham situation as a limiting case of the kinetic theory. Kinetic theory anticipates many of the phenomena (such as start-stop waves) that arise in later work, but probably because the mathematics of working in this framework is fairly laborious, this theory has not been developed any further until recently [16,17].

Instead, in 1971, Payne replaced the assumption of instantaneous adaption in the Lighthill-Whitham theory by an equation for inertia, which is similar to a Navier-Stokes equation [18]. Kühne, in 1984, added a viscosity term and initiated using the methods of nonlinear dynamics for analyzing the equations [19–22]. In a parallel development, Musha and Higuchi proposed the noisy Burgers equation as a model for traffic and backed that up by measurements of the power spectrum of traffic count data [23]. In Sec. III these fluid-dynamical models will be put into a common perspective.

B. Car-following theory

Car-following theory regards traffic from a more microscopic point of view: The behavior of each vehicle is modeled in relation to the vehicle ahead. As the definition indicates, this theory concentrates on single lane situations where a driver reacts to the movements of the vehicle ahead of him. Many car-following models are of the form

$$a(t+T)\propto \frac{v(t)^m}{[\Delta x(t)]^l}\Delta v(t), \qquad (1)$$

where a and v are the acceleration and velocity, respectively, of the car under consideration, Δx is the distance to the car ahead, Δv is the velocity difference to that car, and m and l are constants. T is a delay time between stimulus and response, which summarizes all delay effects such as human reaction time or time the car mechanics needs to react to input.

Other examples for car-following equations are $v(t+T)\propto\Delta x$ [24,25] or $a(t)\propto V[\Delta x(t)]-v(t)$ [26,27], where $V[\Delta x]$ gives a preferred velocity as a function of distance headway. See also [28–30]. Mathematically, parts of this theory are very similar to the treatment of atomic movements in crystals and give results about the stability of chains of cars (‘‘platoons’’) in follow-the-leader situations.

One of the achievements of traffic theory of this period was that relations between car-following models and *static* flow-density relations were derived. Car-following theory will not be treated any further in this paper.

C. Particle hopping models

A more recent addition to the development of vehicular traffic flow theory are particle hopping models. In particle hopping models, a road is represented as a string of cells, which are either empty or occupied by exactly one particle. Movement takes place by hopping between cells. If all particles are updated simultaneously (parallel update, see below), then the particle hopping model treated in this paper formally are also cellular automata (CA).

The technical difference between car-following and CA models for traffic flow is that in the latter, space and time are discrete, whereas in the mathematical treatment of car-following models, they are continuous. *Simulations* of car-following models (e.g., [26–29,107]) discretize time but use continuous space.

Actually, an initial proposition of a CA model for traffic is from Gerlough in 1956 [31] and has been further extended by Cremer and co-workers [32,33]. They implemented fairly sophisticated driving rules and also used single-bit coding with the goal to make the simulation fast enough to be useful for real-time traffic applications. The bit-coded implementation, though, made it too impractical for many traffic applications.

In 1992, CA models for traffic were brought into the statistical physics community. Biham and co-workers used a model with maximum velocity one for one- and for two-dimensional traffic [34]. One-dimensional here refers to roads, etc., and includes multilane traffic. Two-dimensional traffic in the CA context usually means traffic on a two-dimensional grid, as a model for traffic in urban areas. Nagel and Schreckenberg introduced a model with maximum velocity $v_{\max}=5$ for one-dimensional traffic, which compared favorably with real world data [35]. Both approaches were further analyzed and extended in a series of subsequent papers, both for the one-dimensional [28,36–62] (see also [63]) and the two-dimensional (see, e.g., [64–66]) investigations.

That work had two motivations at that time. The primary motivation was again computational speed, but this time to make Monte Carlo analysis possible. The second motivation was to keep the models simple enough to allow analytical treatment. An additional third motivation was added more recently: CA methodology is planned to be used as a high-speed option in traffic projects in Germany [2] and in the United States [1].

From a theoretical point of view, the methodology of particle hopping models lies between fluid-dynamical and car-following theories and helps to clarify the connections between these approaches. One contribution of this paper is to further improve upon the current understanding and to clarify the relations between particle-hopping models and fluid-dynamical models for traffic flow.

III. FLUID-DYNAMICAL MODELS FOR TRAFFIC FLOW

This section reviews fluid-dynamical models for traffic flow. The models can broadly be distinguished by whether they consider the effects of inertia. Models without considering inertia can be derived from the equation of continuity when velocity or current are considered as functions of the density only. Models considering inertia formally are Navier-Stokes equations, with a car-specific force term that takes into account that drivers want to drive at a certain desired speed. If the time constant of this force term is set to zero, i.e., assuming *instantaneous adaption* to the surrounding density, the models revert to the noninertia case.

A. General equations

Papers on traffic flow theory usually start with stating the equations under consideration, without setting them in perspective. I will therefore in this paper attempt a more fundamental approach, similar to conventional fluid dynamics. The precise presentation of most of these equations is necessary anyhow because the particle hopping models presented later relate to these equations.

One might use the standard fluid-dynamical conservation equations for mass and momentum as a starting point for a fluid-dynamical description of traffic:

$$\partial_t \rho + \partial_x(\rho v) = 0 \quad \text{(equation of continuity)} \tag{2}$$

and

$$\frac{dv}{dt} \equiv \partial_t v + v\,\partial_x v = F/m \quad \text{(momentum equation)}, \tag{3}$$

where ρ is the density and v the velocity. d/dt is the individual (Lagrangian) derivative and F is the force acting on mass m. Equation (2) describes mass conservation; Eq. (3) describes the fact that the momentum of a point of mass may only be changed by a force. Obviously, for traffic, F has to include vehicle and driving dynamics.

B. Fluctuations

A standard first step in fluid dynamics [67] is to assume that v and ρ fluctuate statistically around average values $\langle v \rangle$ and $\langle \rho \rangle$, i.e.,

$$v = \langle v \rangle + v', \quad \langle v' \rangle = 0 \tag{4}$$

and

$$\rho = \langle \rho \rangle + \rho', \quad \langle \rho' \rangle = 0. \tag{5}$$

In this case, one only assumes that $\langle v \rangle$ and $\langle \rho \rangle$ fluctuate *slowly* in space and time; for the general subtleties of hydrodynamical theory see, e.g., [68]. Inserting these relations into (2) and (3) and subsequent averaging over the whole equations (e.g., $\langle \partial_x[(\langle \rho \rangle + \rho')(\langle v \rangle + v')] \rangle = \partial_x \langle \rho \rangle \langle v \rangle + \partial_x \langle \rho' v' \rangle$) yields

$$\partial_t \langle \rho \rangle + \partial_x \langle \rho \rangle \langle v \rangle + \partial_x \langle \rho' v' \rangle = 0 \tag{6}$$

and

$$\partial_t \langle v \rangle_L + \langle v \rangle_L \partial_x \langle v \rangle_L + \frac{1}{2} \partial_x \langle v' v' \rangle = \langle F/m \rangle. \tag{7}$$

One often parametrizes averaged fluctuations by the corresponding gradient (see, e.g., [67]) $\langle v' A' \rangle \approx -\alpha \partial_x \langle A \rangle$, which leads to the set of equations

$$\partial_t \rho + \partial_x(\rho v) = D \partial_x^2 \rho,$$
$$\partial_t v + v\,\partial_x v = \nu \partial_x^2 v + F/m, \tag{8}$$

where, according to convention, the averaging angular brackets have been omitted and the diffusion coefficient D as well as the (kinematic) viscosity ν are assumed to be independent of x and t. It should be noted that similar diffusion terms can also be obtained from other arguments. [The idea behind this parametrization is that, if there is more than average of quantity A (i.e., $A > \langle A \rangle$ or $A' \equiv A - \langle A \rangle > 0$) at one location and less than average of quantity A at a neighboring location, then velocity fluctuations represented by v' tend to equilibrate this and that this happens, to first order, linearly in the concentration gradient of A. (Think of A as, say, red color.)]

C. Lighthill-Whitham theory and kinematic waves

If one assumes that the velocity is a function of density only [$v = f(\rho)$], then the momentum equation is no longer necessary. This corresponds to instantaneous adaption; the particles (or cars) carry no memory. Using, without loss of generality, the current $j(\rho) \equiv \rho v(\rho)$ and setting in addition $D = 0$, from (8) one obtains

$$\partial_t \rho + j'(\rho) \partial_x \rho = 0 \tag{9}$$

(the Lighthill-Whitham equation [14]), where $j' = dj/d\rho$. For a review of this theory, see, e.g., [14,69].

The equation can be solved by the ansatz $\rho(x,t) = \rho(x - ct)$ with

$$c = j'(\rho). \tag{10}$$

This allows the solution of the characteristics (see, e.g. [69],): A region with density ρ travels with constant velocity $c = j'(\rho)$ and the resulting straight line in space-time is called characteristic. When $j(\rho)$ is convex, i.e., $j'' < 0$, then for regions of decreasing density [$\rho(x_1) > \rho(x_2)$ for $x_1 < x_2$] the characteristics separate from each other. In regions of *increasing* density, the characteristics come closer and closer together. When two characteristics touch each other, a density discontinuity appears at this place (a front), which moves with velocity

$$c=\frac{j(x_2)-j(x_1)}{\rho(x_2)-\rho(x_1)}=\frac{\Delta j}{\Delta\rho}. \tag{11}$$

Note that formally the fluid-dynamical description has broken down here because both ρ and j are no longer continuous functions of x.

An illustrative example is a queue, such as at a red light. When the light turns green, the outflow front quickly smoothes out, whereas the inflow front remains steep.

Note that usually at maximum flow $c=j'=0$. Structures that operate at maximum flow do not move in space.

Leibig [70] gives results of how a random initial distribution of density steps in a closed system evolves towards two single steps according to the Lighthill-Whitham theory.

D. Lighthill-Whitham theory with dissipation

Adding dissipation to the Lighthill-Whitham equation leads to

$$\partial_t\rho+j'(\rho)\partial_x\rho=D\partial_x^2\rho. \tag{12}$$

The solution of this equation is again a nondispersive wave with phase and group velocity j'. The difference is that D introduces dissipation (damping) of the wave: The amplitude decays as e^{-Dk^2}, where k is the wave number. This reflects the intuitively reasonable effect that traffic jams should tend to dissolve under homogeneous and stationary conditions.

E. The nonlinear diffusion (Burgers) equation

For a further development, $j(\rho)$ has to be specified. Since we are mostly interested in the behavior of traffic near maximum throughput, we start by choosing the simplest mathematical form that yields a ''well-behaved'' maximum

$$j(\rho)=v_{\max}\rho(1-\rho), \tag{13}$$

which, in traffic science, is called the Greenshields model (see [10]). $v_{\max}$ is, in principle, a free parameter, but it has an interpretation as the maximum average velocity for $\rho\to 0$. Mathematicians would set $v_{\max}=1$; traffic scientists use $1-\rho/\rho_{\rm jam}$ for the term in parenthesis. $\rho_{\rm jam}$ is the density of vehicles in a jam. The maximum current $j_{\max}$ is reached at $\rho(j_{\max})=1/2$.

Substituting (13) into (12) yields

$$\partial_t\rho+v_{\max}\partial_x\rho-2v_{\max}\rho\partial_x\rho=D\partial_x^2\rho. \tag{14}$$

Musha and Higuchi [23] have shown that by introducing a linear transformation of variables

$$x=v_{\max}t'-x', \quad t=t', \tag{15}$$

one obtains

$$\partial_{t'}\rho+2v_{\max}\rho\partial_{x'}\rho=D\partial_{x'}^2\rho, \tag{16}$$

which is the (deterministic) Burgers equation [71].

The transformation (15) does two things.

(i) Transformation to a coordinate system that is moving with uniform velocity $v_{\max}$, that is, vehicles with $v_{\max}$ do not move at all in this new coordinate system and slower vehicles move backward (i.e., to the left).

(ii) A reversal of direction, i.e., the vehicles that are moving backwards after part (i) of the transformation now move to the right. Note that this causes a change of sign before the nonlinear term, which does not have any explanatory value except that it brings Eq. (16) *exactly* to the form treated by Burgers.

This equation has been investigated in great detail by Burgers [71] as the simplest non-linear diffusion equation. The stationary solution is a uniform density $\rho(x,t)=$ const. A single disturbance from this state evolves over time into a characteristic triangular structure with amplitude $\sim t^{-1/2}$, width $\sim t^{1/2}$, bent to the right such that the right-hand side of the disturbance becomes discontinuous, and moving to the right with velocity $c=j'=2\rho v_{\max}$.

When interpreting this for traffic jams, one has to retransform the coordinates. Jams can then move *both* to the left or to the right (with velocities between $v_{\max}$ and $-v_{\max}$) and the discontinuous front develops at the inflow side of the jam, i.e., where the vehicles enter the jam. One sees that this solution is just the solution of the characteristics, with a dissipating diffusion term added, as should be expected because of $D>0$.

Some other versions of the Burgers equation have been investigated thoroughly [72–74]. Of interest in the context of this paper are the following.

Noisy Burgers equation. Adding a Gaussian noise term η to the equation [i.e. $\langle\eta(x,t)\eta(x',t')\rangle=\eta_0\delta(x-x')\delta(t-t')$] leads to the noisy Burgers equation

$$\partial_t\rho+2v_{\max}\rho\partial_x\rho=D\partial_x^2\rho+\eta. \tag{17}$$

This equation no longer converges towards a homogeneous state.

Generalized Burgers equation. The nonlinearity of the Burgers equation can be generalized

$$\partial_t\rho=\sum_\beta b_\beta\partial_x\rho^\beta+D\partial_x^2\rho. \tag{18}$$

Generalized Burgers equations with arbitrary β have been investigated [73,72].

F. Including momentum

The equations so far do not explain the spontaneous phase separation into relatively free and rather dense regions of vehicles, which is observed in real traffic. To obtain this, one has to include the effect of momentum: One can neither accelerate instantaneously to a desired speed nor slow down without delay. It becomes necessary to include the momentum equation. Here one has to specify the force term F/m, which describes acceleration and slowing down. At least two properties are usually incorporated, which are called the ''relaxation term'' and the ''interaction term.''

A first-order approximation for the relaxation term is [19,18]

$$\frac{1}{\tau}[V(\rho)-v], \tag{19}$$

where $V(\rho)$ is the desired average speed as a function of density and τ is a relaxation time. This choice yields exponential relaxation towards the desired speed. The function $V(\rho)$ has to be specified externally, for example, from measurements.

A commonly used interaction term [75–77,19,18] is

$$-\frac{c_0^2}{\rho}\partial_x\rho. \tag{20}$$

The meaning is that one tends to reduce speed when the density increases, even when the local density is still consistent with the current speed.

A more formal possible derivation of the interaction term is as follows. (I got the idea for this argument from B.S. Kerner.) In real traffic, the relaxation term actually is asymmetric with respect to the vehicle position, e.g., say, $\dot{v}(x)=1/\tau[V(\Delta x)-v]$ (see car-following section), where Δx is the front-buffer-to-front-buffer distance to the next vehicle ahead.

After approximating $V(\Delta x)$ by $V(\rho(x+\Delta x/2))$ and then Taylor expanding, one obtains

$$\frac{1}{\tau}[V(\Delta x)-v]\approx\frac{1}{\tau}[V(\rho(x))-v]+\frac{1}{\tau}\Delta x V'(\rho(x))\partial_x\rho, \tag{21}$$

thus obtaining a formal justification of the interaction term $\sim\partial_x\rho$ out of the asymmetry of the relaxation term. c_0 is treated as constant; in traffic, a typical value for c_0 is 15 km/h [20].

Formally, the interaction term is similar to the pressure term of compressible flow $-(1/\rho)\partial_x p$, where p is the pressure. Assuming an ideal gas ($p=\rho RT$) and isothermic behavior $T=$const, one obtains waves similar to sound waves as a solution of the linearized equations. (True sound waves, though, would assume the gas to behave adiabatically, i.e., $p\propto\rho^\kappa$.) This leads to Eq. (20), where c_0 is the speed of the "sound" waves. (See below for a short discussion.) Note that sound waves move in *both* directions from a disturbance, which means that sound waves alone are not a good explanation for freeway start-stop waves, contrary to what is sometimes written [78].

Taking all this together, a possible momentum equation for traffic therefore is [19]

$$\partial_t v+v\,\partial_x v=-\frac{c_0^2}{\rho}\partial_x\rho+\frac{1}{\tau}[V(\rho)-v]+\nu\partial_x^2 v. \tag{22}$$

Since one now has two variables, one also needs an equation of continuity to close the system:

$$\partial_t\rho+\partial_x(\rho v)=D\partial_x^2\rho. \tag{23}$$

Usually, D is set to zero.

For this equation, the homogeneous solution $(v,\rho)\equiv(v_0,\rho_0)$ is unstable for densities near maximum flow for a suitable choice of parameters. Using the methods of nonlinear dynamics, Kühne and co-workers [19,22,21] went beyond linear stability analysis (see also [79,80]). One finds a multitude of stable or unstable fixed points and limit cycles, which suggest that traffic near maximum flow operates on a strange attractor. This can lead to quasi-periodic behavior, exactly as is observed in traffic measurements. Earlier work [75,18] has analyzed the same equation without viscosity ($\nu=0$).

G. Discussion of fluid-dynamical approaches

Fluid-dynamical models have been used in traffic science for a long time, with considerable success. But they have shortcomings. Some of the major points are the following.

(i) One has to give externally the relation between speed or current and density. This is unsatisfying in terms of the development of a theory. But an even more intricate problem is that there is no agreement on a functional form of the speed-density relation; it is even under discussion if this relation is at all continuous [13,81].

(ii) Microscopically, temperature parametrizes the random fluctuations of particles around their mean speed: $T\propto\langle v^2\rangle-\langle v\rangle^2$. For gases, fluctuations, and therefore temperature, increase with density. For granular media such as vehicular traffic or sand, fluctuations *decrease* with density (i.e., inside a jam); it has been claimed that exactly this inverse temperature effect is responsible for clustering [82]. In this way, assuming isothermic instead of adiabatic behavior as done for the momentum equation seems only half the way one has to go. Helbing [83] discusses this further.

(iii) Helbing [83] also discusses the effect of excluded volume to take into account the spatial extension of vehicles.

(iv) Daganzo [84] claims that *all second-order* fluid-dynamical models produce unrealistic behavior (such as backward moving vehicles caused by a diffusion term) and are therefore unsuitable for traffic science.

Nonetheless, fluid-dynamical approaches [76,19,22,21] give systematic insight into traffic dynamics near maximum flow beyond simple extrapolation of light and dense traffic results. These results will be further discussed near the end of this paper.

IV. DEFINITIONS OF PARTICLE HOPPING MODELS

This section defines several particle hopping models that are candidate models for traffic. They all are commonly defined on a lattice of, say, length L, where L is the number of sites. Each site can be either empty or occupied by exactly one particle. Also, in all models particles can only move in one direction. The number of particles N is conserved except at the boundaries. For traffic, particles model cars.

A. The stochastic traffic cellular automaton

The stochastic traffic cellular automaton (STCA), which has been treated in a series of papers [43–56], is defined as follows. Each particle (car) can have an integer velocity between 0 and $v_{\max}$. The complete configuration at time step t is stored and the configuration at time step $t+1$ is computed from that, i.e., using a parallel or synchronous update. All cars (particles) execute in parallel the following steps.

(i) Let g (gap) equal the number of empty sites ahead.

(ii) If $v>g$ (too fast), then slow down to $v:=g$ (rule 1); otherwise if $(v<g)$ (enough headway) *and* $v<v_{max}$, then accelerate by one: $v:=v+1$ (rule 2).

(iii) Randomization: If after the above steps the velocity is larger than zero $(v>0)$, then, with probability p, reduce v by one (rule 3).

(iv) Particle propagation: Each particle moves v sites ahead (rule 4).

The randomization incorporates three different properties of human driving into one computational operation: fluctuations at maximum speed, overreactions at braking, and retarded (noisy) acceleration. Note that, because of integer arithmetic, conditions such as $v>g$ and $v\geqslant g+1$ are equivalent.

When the maximum velocity of this model is set to one $(v_{max}=1)$, then the model becomes much simpler: Each particle executes the following in parallel: If a site ahead is free, move, *with probability* $1-p$, to that site. Since the STCA shows different behavior for $v_{max}\geqslant 2$ than for $v_{max}=1$, we will distinguish them as STCA/1 and STCA/2, respectively.

Due to the given discretization of space and time, proper units are often omitted in the context of particle hopping or cellular automata models. Proper units here would be $[g]$ equal to the number of cells, $[v]$ equal to the number of cells per time step, $[t]$ equal to the number of time steps, etc. For that reason, it is possible to write something such as $v<g$, which properly would have to be $v<g$/(time step). Note that one still needs conversion factors to convert, say, velocity from the particle hopping model to a real world velocity, e.g., given in km/h. One should note, though, that every computer program does such a thing. Numbers in computer programs are always unitless and a proper conversion to real world numbers has to be put in by the program designer.

B. The cruise control limit of the STCA (STCA-CC)

In the so-called cruise control limit of the STCA [51], fluctuations at free driving, i.e., at maximum speed and undisturbed by other cars, are set to zero. Algorithmically, the velocity update (rules 1–3) of the STCA are replaced by the following. For all cars, do the following in parallel.

(i) A vehicle is stationary when it travels at maximum velocity v_{max} and has free headway $g\geqslant v_{max}$. Such a vehicle just maintains its velocity.

(ii) Otherwise (i.e., if a vehicle is not stationary) the standard rules 1–3 of the STCA are applied.

Both acceleration and braking still have a stochastic component.

C. The deterministic limit of the STCA (CA-189)

One can take the deterministic limit of the STCA by setting the randomization probability p equal to zero, which just amounts to skipping the randomization step. It turns out that, when using a maximum velocity $v_{max}=1$, this is equivalent [72] to the cellular automaton rule 184 in Wolfram's notation [85], which is why I will use the notation CA-184/1 and CA-184/2.

Much work using CA models for traffic is based on this model. Biham and co-workers [34] have introduced it for traffic flow, with $v_{max}=1$. Other authors base further results on it [28,36,37,39,40,43]. Some [28,40] also use it with v_{max} larger than one. It is also the basis of the two-dimensional CA models for traffic (e.g., [64–66]).

D. The cruise control version for the CA-184 (CA-184-CC)

Takayasu and Takayasu [42] introduced a different CA model that is effectively equivalent to a deterministic cruise control situation for CA-184/1. This may not be obvious from the rules, but it will become clear from the dynamic behavior summarized later. Since they use only maximum velocity $v_{max}=1$, the rules are short. For all particles do the following in parallel.

(i) If $v=1$ and the site ahead is free $(g\geqslant 1)$, then move one site ahead.

(ii) A particle at rest $(v=0)$ can move only when $g\geqslant 2$.

Generalizations to maximum velocity larger than one are straightforward, but do not seem to lead to additional insight.

E. The asymmetric stochastic exclusion process

The probably most-investigated particle hopping model is the asymmetric stochastic exclusion process (ASEP). Its behavior is defined as follows.

(i) Pick one particle randomly (rule 1).

(ii) If the site to the right is free, move the particle to that site (rule 2).

The ASEP is closely related to CA-184/1 and STCA/1 (i.e., both with maximum velocity one). The difference actually only is in the manner in which sites are updated. CA-184 and STCA update all sites synchronously, whereas ASEP uses a random serial sequence.

In order to compare the ASEP with the other, synchronously updated models, one has to note that, in the ASEP, *on average* each particle is updated once after N single-particle updates. A time step (also called update step or iteration) in the ASEP is therefore completed after N single-particle updates (which is equal to N attempted hops).

It has been noted in Ref. [72] that changing the update from asynchronous to synchronous, i.e., going from ASEP to CA-184/1, changes the dynamics considerably. In this paper, I will in addition show that reintroducing the randomness via the randomization (rule 4) in the STCA again leads to different results.

A systematic way of reducing the noise for the ASEP could be done using techniques described by Wolf and Kertesz [86], i.e., by putting a counter on each particle and moving it only after k trials. For large k it becomes more and more improbable that one particle is moved twice while a neighboring particle is not moved at all during that time. Taking the limit $k\rightarrow\infty$ then reduces the ASEP to the CA-184 process in a smooth way.

One can also define higher velocities for the ASEP by simply replacing the ASEP rule 2 by STCA/2 rules 1, 2, and 4. In such a case, each particle has to remember its velocity v from the last move.

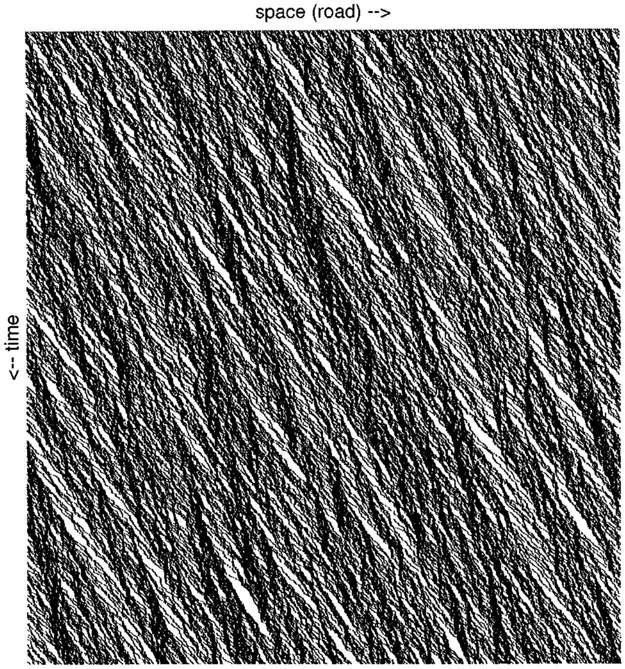

FIG. 1. Space-time plot for random sequential update, $v_{\max}=1$ (ASEP/1) and $\rho=0.3$. Clearly, the kinematic waves are moving forward. For $\rho>1/2$, the kinematic waves would be moving backward and the plot would look similar to Fig. 2.

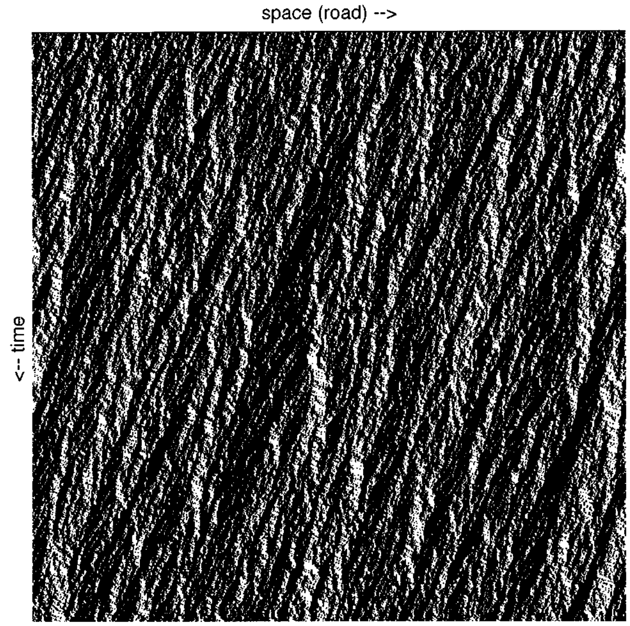

FIG. 2. Space-time plot for random sequential update, $v_{\max}=5$ (ASEP/5) and $\rho=0.3$, showing that the higher maximum velocity does not lead to a different appearance as long as one uses the random sequential update.

V. PARTICLE HOPPING MODELS, FLUID DYNAMICS, AND CRITICAL EXPONENTS

Both for the ASEP/1 and for the CA-184/1, fluid-dynamical limits and critical exponents are well known (see, e.g. [74,72,87,73],). The most straightforward way to put the concept of critical exponents into the context of traffic flow is to consider ''disturbances'' (i.e., jams) of length x and ask for the time t to dissolve them. For example, one would intuitively assume that a queue of length x at a traffic light that just turned green would need a time t proportional to x until everybody is in full motion. By this argument, the dynamic exponent z, defined by $t\sim x^z$, should be one.

Yet, there may be more complicated cases. Imagine again a queue at a traffic light just turned green, but this time there is also some fairly high inflow at the end of the queue. The jam queue itself will start moving backward, clearing its initial position in time $t\sim x$. However, the dissolving of the jam itself may be governed by different rules. An example for this will be given in the following.

A. ASEP/1

It can be shown that the classic ASEP corresponds to the noisy Burgers equation (see, e.g., [72,73]). More precisely, the hydrodynamic limit of the particle process is a diffusion equation $\partial_t\rho+\partial_x j=D\partial_x^2\rho+\eta$ with a current [72,88,89] of $j=\rho(1-\rho)$. This yields

$$\partial_t\rho+\partial_x\rho-\partial_x\rho^2=D\partial_x^2\rho+\eta, \qquad (24)$$

which is exactly the Lighthill-Whitham-Greenshields case with noise and diffusion described earlier. In other words, the ASEP/1 particle hopping process and the Lighthill-Whitham theory (plus noise plus diffusion), specialized to the case of the Greenshields flow-density relation, describe the *same* behavior.

In the steady state, this model shows kinematic waves (small jams), which are produced by the noise and damped by diffusion (Fig. 1). These nondispersive waves move forward (wave velocity $c=j'=1-2\rho>0$) for $\rho<1/2$ and backward ($c<0$) for $\rho>1/2$ (Fig. 2). At $\rho=1/2$, the wave velocity is exactly zero ($c=0$) and this is the point of maximum throughput [90]. If traffic were modeled by the ASEP, then one could detect maximum traffic flow by standing on a bridge: Jam waves moving in flow direction indicate too low density (cf. Fig. 1) and jam waves moving against the flow direction indicate too high density.

The ASEP is one of the cases where clearing a site follows a different exponent than dissolving a disturbance. (Note that, technically, all these remarks are only valid for *small* disturbances. When the system is not close to the steady state, one sees transient behavior that may be different [72].) As long as $\rho\neq 1/2$, a disturbance of size x moves with speed $c\neq 0$ and therefore clears the initial site in time $t\sim cx\sim x^1$, i.e., with dynamical exponent $z=1$. In order to see how the disturbance itself dissolves, one transforms into the coordinate system of the wave velocity. One conventionally does that by first separating between the average density $\langle\rho\rangle_L$ and the fluctuations ρ'. By inserting $\rho=\langle\rho\rangle_L+\rho'$ one obtains

$$\partial_t\rho'+(1-2\langle\rho\rangle_L)\partial_x\rho'-2\rho'\partial_x\rho'=D\partial_x^2\rho'+\eta. \qquad (25)$$

When transforming this into the moving coordinate system $x'=x+(1-2\langle\rho\rangle_L)t$, one obtains

$$\partial_t\rho'-2\rho'\partial_x'\rho'=D\partial_x'^2\rho'+\eta, \qquad (26)$$

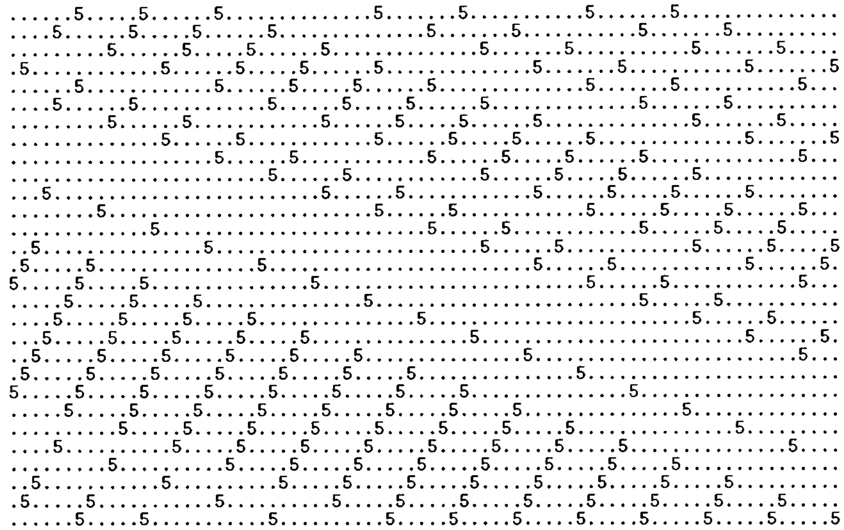

FIG. 3. Space-time plot for CA-184/5 and subcritical density.

which is the classic noisy Burgers equation.

Note that this transformation is different from the Musha transformation Eq. (15). As a formality, we do not change the sign of the x direction here and in consequence not the sign of the nonlinear term. More important, we transform into a coordinate system moving with the speed of the *waves* and this time find that *the fluctuations* of the system also obey the Burgers equation.

For this equation it is well known that the dynamical exponent is $z=3/2$ (the Kardar-Parisi-Zhang exponent). In other words, in the original coordinate system a disturbance four times as big as another one, $x'=4x$, needs $t'\sim x'=4x\sim 4t$, i.e., four times as much time to clear the site, but $t'\sim x'^{3/2}=(4x)^{3/2}\sim 8t$, i.e., 8 times as much time until the jam structure itself is no longer visible in the noise. A precise treatment of this uses, e.g., correlations between tagged particles [74].

The drawback of this model with respect to traffic flow is that it has neither a regime of laminar flow nor ''real,'' big jams. Because of the random sequential update, vehicles with average speed $\overline{v}$ fluctuate severely around their average position given by $\overline{v}t$. As a result, they always ''collide'' with their neighbors, even at very low densities, leading to ''minijams'' everywhere. This is clearly unrealistic for light traffic.

Actually, this fact is also visible in the speed-density relation. Using the Greenshields flow-density relation, one obtains

$$v=\frac{j}{\rho}\propto 1-\rho. \tag{27}$$

This is in contrast to the observed result that, at low densities, speed is nearly independent of density (practically no interaction between vehicles).

B. ASEP/2

Judging from space-time plots (see Figs. 1 and 2), changing the maximum velocity in the update from $v_{\max}=1$ to $v_{\max}\geqslant 2$ does not change the universality class [54]. It skews the flow-density relation towards lower densities, but does not lead to other phenomenological behavior.

C. CA-184

Using a maximum velocity higher than one does not change the general behavior of CA-184. It therefore makes sense to directly discuss the general case.

As explained above, the CA-184/1 is the deterministic counterpart of the ASEP/1. But taking away the noise from the particle update completely changes the universality class (i.e., the exponent z) [72]. The model now corresponds to the nondiffusive, nonnoisy equation of continuity

$$\partial_t\rho + j'\,\partial_x\rho = 0, \tag{28}$$

with the (except at $\rho=\rho_{j_{\max}}$) linear flow

$$j'=\frac{dj}{d\rho}=\begin{cases} v_{\max} & \text{for } \rho<\rho_{j_{\max}} \\ -1 & \text{for } \rho>\rho_{j_{\max}}. \end{cases} \tag{29}$$

The intersection point of the fundamental diagram divides two phenomenological regimes: light traffic ($\rho<\rho_{j_{\max}}$) and dense traffic ($\rho>\rho_{j_{\max}}$).

A typical situation for light traffic is shown in Fig. 3 (with $v_{\max}=5$). After starting from a random initial condition, the traffic relaxes to a steady state, where the whole pattern just moves $v_{\max}=5$ positions to the right in each iteration. Cars clearly have a tendency of keeping a gap of $\geqslant v_{\max}=5$ between each other. As a result, the current j in this regime is

$$j_< = \rho v_{\max}. \tag{30}$$

The velocity of the kinematic waves in this regime is $c_< = j'_< = v_{\max}$. This means that disturbances, such as holes, just move with the traffic, as can also be seen in Fig. 3.

Dense traffic is different (Fig. 4). Again starting from a random initial configuration, the simulation relaxes to a steady state where the whole pattern moves one position to the *left* in each iteration. Note that cars still move to the right; if one follows the trajectory of one individual vehicle,

```
..3...1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2.
.3...1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..
3...1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3
...1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3.
..1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3..
.1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...
1.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4
.2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4.
2..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4..
..3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4...
.3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....
3...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5
...4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.
..4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5..
.4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5...
4....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5....
....5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....
...5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5
..5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.
.5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5..
5.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5...
.....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5....
....5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....
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..5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5.
.5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5..
5.....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5...
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....5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5.....
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..5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5.....5.
.5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5.....5..
5.....5.....5.....5.....1.01.2..3...3...1.1.2..3...4....5.....5.....5.....5...
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FIG. 4. Space-time plot for CA-184/5 and supercritical density.

for this car regions of relatively free movement are alternating with regions of high density and slow speed. Although in a too static way, this captures some of the features of start-stop traffic. The average speed in the steady state equals the number of empty sites divided by the number of particles: $\langle v \rangle_L = (L-N)/N$; the current is $j_> = \rho \langle v \rangle_L$ or, with $\rho = N/L$,

$$j_> = 1 - \rho. \tag{31}$$

This straight line intersects with the one from light traffic at $\rho = 1/(1+v_{\max})$, which is therefore the density corresponding to maximum throughput $j_{\max} = v_{\max}/(1+v_{\max})$.

The velocity of the kinematic waves in the dense regime is $j'_> = -1$, which corresponds to the backward moving pattern in Fig. 4.

Since the second term of Eq. (28) [with (29)] is (except at $\rho = \rho_{j_{\max}}$) linear in the density, these are linear Burgers equations and the dynamic exponent z is equal to 1 [72]. More precisely, in terms of traffic the following happens. The outflow of a jam in this model always operates at flow $j_{\rm out} = j_{\max}$ and density $\rho_{\rm out} = \rho_{j_{\max}}$. The time t until a jam of length x dissolves therefore obeys the average relation $t \propto x/(j_{\max} - j_{\rm in})$, where $j_{\rm in}$ is the average inflow to the jam. Since $j \propto \rho$ for $\rho \leqslant \rho_{j_{\max}}$, one can write that as

$$t \propto \frac{x}{\rho_{j_{\max}} - \rho(j_{\rm in})}. \tag{32}$$

This means that for $\rho < \rho_{j_{\max}}$, the critical exponent z is indeed one, but at $\rho = \rho_{j_{\max}}$, t diverges. This effect is also visible when disturbing the system from its stationary state [28]: The transient time $t_{\rm trans}$ until the system is again stationary scales as

$$t_{\rm trans} \sim \frac{1}{\rho_{j_{\max}} - \rho}. \tag{33}$$

The scaling law (33) is actually also true for $\rho > \rho_{j_{\max}}$, albeit for a different reason with a slightly more complicated phenomenology. See [28] for more details.

Two observations are important at this point.

(i) Many papers in the physics literature [28,34,36,37,39–41, 43] use this model for their investigations. Also the two-dimensional grid models (see, e.g., [64–66]) essentially use this model for the one-dimensional part of their movements, although the two-dimensional interactions seem to change the flow-density relationship [104]. The CA-184 model lacks at least two features that are, as I will argue later, important with respect to reality: (a) The first is bistability: Laminar flow above a certain density becomes instable, but can exist for long times. CA-184 does not display this bistability. (b) The other is stochasticity: CA-184 is completely deterministic, i.e., a certain initial condition always leads to the same dynamics. Real traffic, however, is stochastic, that is, even identical initial conditions will lead to different outcomes, and a model should be capable of calculating some distribution of outcomes (by using different random seeds).

(ii) The so-called cell transmission model [91], which has been proposed for traffic applications, technically is a discretization of the Lighthill-Whitham theory. It turns out that this model is similar to Eq. (28) with (29), especially with respect to the range of physical phenomena that are represented. The only difference is that the j-ρ relation of Ref. [91] has a flat portion at maximum flow instead of the single peak of Eq. (29). That means that in the cell transmission model low-density and high-density traffic behave similarly to CA-184, but traffic at capacity has a regime where waves do not move at all.

Using other j-ρ relations in discretized Lighthill-Whitham models (e.g., [92,79]) will lead to other relations for the wave speeds, but the range of physical phenomena (backward or forward moving waves) that can be represented will always resemble CA-184; in particular, neither the bistability nor the stochasticity can be represented.

D. CA-184-CC

No fluid-dynamical limits for the other particle hopping models are known. Yet, results for the jam dynamics for the cruise control situations [42,51] offer valuable insights.

The important feature of the cruise control version of CA-184 is a bistability [42]. Using $v_{\max}=1$ in this section ($v_{\max}>1$ does not seem to offer additional insight), this bistability occurs between two densities, i.e., for $\rho_{c1}<\langle\rho\rangle_L<\rho_{c2}=1/2$, where $\langle\rho\rangle_L:=N/L$, and, for $v_{\max}=1$, $\rho_{c1}=1/3$, and $\rho_{c2}=1/2$. $\langle\ \rangle_L$ means the average over the whole (closed) system of length L. In this range, some initial conditions lead to laminar flow, but others lead to traffic including jams. Takayasu and Takayasu determined that density-velocity relations converge to two types, which depend upon initial conditions.

(i) Starting with maximally spaced particles and initial velocity $v=1$, one finds stable configurations with flow $\langle j\rangle_L=\langle\rho\rangle_L v_{\max}=\langle\rho\rangle_L$ for low densities $\langle\rho\rangle_L\leqslant 1/2=:\rho_{c2}$. However, for high densities $\langle\rho\rangle_L>\rho_{c2}$, a jam phase appears for *all* initial conditions since not all particles can keep $g\geqslant 1$. Once a jam has been created, all particles in the outflow of this jam have $g=2$. For $t\to\infty$, this dynamics reorganizes the system into jammed regions with density $\rho=1$ and zero current $j=0$ and laminar outflow regions with $\rho_{\text{out}}=1/3$ and $j_{\text{out}}=1/3$. Simple geometric arguments then lead, for the whole system, to $\langle j\rangle_L=(1-\langle\rho\rangle_L)/2$ and $\langle v\rangle_L=(1/\langle\rho\rangle_L-1)/2$.

(ii) Starting, however, with an initial condition where all particles are clustered in a jam, this jam is only sorted out up to $\langle\rho\rangle_L\leqslant 1/3=:\rho_{c1}$, leading to $\langle j\rangle_L=\langle\rho\rangle_L$ and $\langle v\rangle_L=1$. For $\langle\rho\rangle_L>\rho_{c1}$, the initial jam survives forever, yielding $\langle j\rangle_L=(1-\langle\rho\rangle_L)/2$ and $\langle v\rangle_L=(1/\langle\rho\rangle_L-1)/2$. One observes that, for $\rho_{c1}<\langle\rho\rangle_L<\rho_{c2}$, this initial condition leads to a different final flow state than the initial conditions in (i). Note that ρ_{c1} is equal to the outflow density ρ_{out}.

Starting from an arbitrary initial condition, the density-velocity relation converges to one of the above two types.

Note that up until this section, all relations between j, v, and ρ were also locally correct, which is why averaging angular brackets were omitted. Now, this is no longer true. For example, densities slightly above ρ_{c2} do not really exist on a local level; they are only possible as a global composition of regions with local densities $\rho=\rho_{c1}$ plus others with local densities $\rho=1$.

Since the model is deterministic, one can calculate the behavior from the initial conditions. For any particle i with initial velocity $v=0$ one can determine the influence i has on particles ''behind it'' $(i+1,i+2,\dots)$. For particle $i+k$ to be the first one not to be involved in the jam caused by i, one needs the average gap between i and k to be larger than two. This corresponds to a density between i and k of $\rho_{ik}<1/(g+1)=1/3=\rho_{c1}$. The sequence $(g_{i+j})_j$ describes a random walk, which is positively (negatively) biased for $\rho>\rho_{c1}$ $(\rho<\rho_{c1})$ and unbiased at the critical point $\rho=\rho_c=\rho_{c1}$ [42].

E. STCA-CC/1

The cruise control limit of the STCA combines properties of the CA-184 and the full STCA. Since the STCA-CC has no fluctuations at free driving, the maximum flow one can reach is with all cars at maximum speed and $g=v_{\max}$. Therefore, one *can* manually achieve flows that follow, for $\rho\leqslant\rho_{c2}$, the same j-ρ relationship as the CA-184, where ρ_{c2} now denotes the density of maximum flow of the deterministic model CA-184, i.e., $\rho_{c2}=1/(v_{\max}+1)$.

Above a certain ρ_{c1}, these flows are unstable to small local perturbations. This density will turn out to be a ''critical'' density; for that reason I will use the notation $\rho_c\equiv\rho_{c1}$. Many different choices for the local perturbation give rise to the same large-scale behavior. The perturbed car eventually reaccelerates to maximum velocity. In the meantime, though, a following car may have come too close to the disturbed car and has to slow down. This initiates a chain reaction—an emergent traffic jam.

It is straightforward to see [51] that $n(t)$, the number of cars in the jam, follows a usually biased, absorbing random walk, where $n(t)=0$ is the absorbing state (jam dissolved): Every time a new car arrives at the end of the jam, $n(t)$ increases by one, and this happens with probability j_{in}, which is the inflow rate. Every time a car leaves the jam at the outflow side, $n(t)$ decreases by one, and this happens with probability j_{out}. When $j_{\text{in}}=j_{\text{out}}$, $n(t)$ follows an unbiased absorbing random walk. $j_{\text{in}}\neq j_{\text{out}}$ introduces a bias or drift term proportional to $(j_{\text{in}}-j_{\text{out}})t$.

This picture is consistent with Takayasu and Takayasu's observations for the CA-184-CC model. The main difference is that now *both* the inflow gaps and the outflow gaps form a random sequence. Another difference is conceptual: Takayasu and Takayasu have looked at the transient time starting from initial conditions rather than looking at jams starting from *a single disturbance*. The latter approach [51] leads to a cleaner picture of the traffic jam dynamics because it concentrates on the transition from laminar to start-stop traffic, which is observed in real traffic.

The statistics of such absorbing random walks can be calculated exactly. For the unbiased case one finds that

$$\langle n(t)\rangle\sim t^{\eta},\quad P_{\text{surv}}(t)\sim t^{-\delta},\quad \langle w(t)\rangle_{\text{surv}}\sim t^{\eta+\delta}, \tag{34}$$

where P_{surv} is the survival probability of a jam until time t and $w(t)$ means the width of the jam, i.e., the distance between the leftmost and the rightmost car in the jam. $\langle\ \rangle$ means the ensemble average over *all* jams that have been initiated and $\langle\ \rangle_{\text{surv}}$ means the ensemble average over *surviving* jams. For the critical exponents, one finds as well from theory as from numerical simulations $\delta=1/2$ and $\eta=0$. $\eta=0$ reconfirms that, at the critical density ρ_c, jams in the average *barely survive* (unbiased random walk).

If one now uses j_{in} as the order parameter and, say, $P_{\text{surv}}(t)$ as the control parameter, then we have a second-order phase transition, where

$$\begin{aligned} P_{\text{surv}}(t) &= 0 \quad \text{for } j_{\text{in}}<j_{\text{out}},\quad t\to\infty \\ &\sim t^{-\delta} \quad \text{for } j_{\text{in}}=j_{\text{out}},\quad t\to\infty, \\ &= \text{const} \quad \text{for } j_{\text{in}}>j_{\text{out}},\quad t\to\infty. \end{aligned} \tag{35}$$

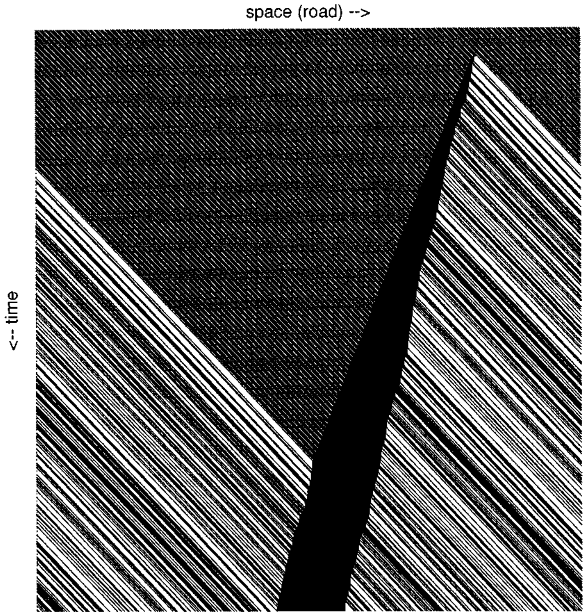

FIG. 5. Space-time plot for STCA-CC/1, at supercritical density, with one disturbance. The jam first grows according to $n(t) \sim (j_{\text{in}} - j_{\text{out}})t$. Eventually, via the periodic boundary conditions, the outflow reaches the jam as inflow and $n(t)$ follows a random walk (apart from finite size effects).

For that reason, we call $j_c := j_{\text{out}}$ the critical flow and the associated density $\rho_c := \rho(j_c)$ the critical density.

It is important to note that $j_{\text{in}} > j_{\text{out}}$ as a stable, long-time state is only possible with the particular definition of the cruise control limit and the use of an open system. If one would use a closed system (periodic boundary conditions, i.e., traffic in a loop), the outflow of the jam would eventually recirculate around loop and become the inflow of the jam (see Fig. 5), leading to the situation $j_{\text{in}} = j_{\text{out}}$; if one relaxes the cruise control assumption, eventually other jams would form upstream of the one under consideration and the outflow of these jams would eventually be the inflow of the jam under consideration, again leading to $j_{\text{in}} = j_{\text{out}}$.

Deviations from the cruise control limit will be addressed later; let us now consider a *closed* system.

(i) For $\rho < \rho_c$ and arbitrary initial conditions, jams are ultimately sorted out. Then, every car has velocity $v = v_{\max}$ and $g \geqslant v_{\max}$, is thus in the free driving regime as defined above.

(ii) For $\rho_c \leqslant \rho \leqslant \rho_{c2}$, the long-time behavior depends on the initial conditions. For example, even in the extreme case of $\rho = \rho_{c2}$, the state where every car has velocity $v = v_{\max}$ and $g = v_{\max}$ is stable and results in a flow of $j = v_{\max}\rho_{c2}$. However, most other initial configurations will lead to jams and for the limit of infinite system size, at least one of them never sorts out.

(iii) For $\rho > \rho_{c2}$, all initial conditions lead to jams.

Note that this is again consistent with the results of Takayasu and Takayasu for the CA-184-CC system [42].

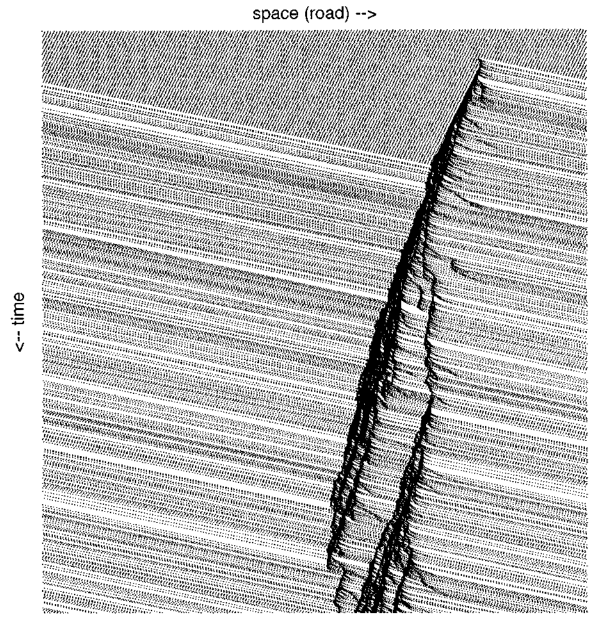

FIG. 6. Space-time plot for parallel update (cruise control limit), $v_{\max} = 5$ and $\rho = 0.09$, i.e., slightly above critical. The flow is started in a deterministic, supercritical configuration, but from a single disturbance separates into a jam and a region of exactly critical density. This is phenomenologically the same plot as Fig. 5, except that $v_{\max} = 5$.

F. STCA-CC/2

Replacing the maximum velocity $v_{\max} = 1$ by $v_{\max} \geqslant 2$ (Fig. 6) does not change the critical behavior, but it adds a complication [51]. Now, jam clusters can branch, with large jam-free holes in between branches of the jam. As a result, space-time plots of such jams now appear to show fractal properties and in simulations at the critical density $w(t)$ no longer follows a clean scaling law, whereas $n(t)$ and P_{surv} still do.

The explanation for this is that the holes in the jam are large enough to cause logarithmic corrections to the width, but not large enough to make it completely fractal. More precisely, the hole size distribution $P_h(x)$, i.e., the probability to find a hole of size x in a given equal time cut (jam configuration), scales as

$$P_h(x) \sim x^{-\tau_h}, \tag{36}$$

where both from a theoretical argument and from simulations $\tau_h = 2$. Yet, it is known that for $\tau_h \leqslant 2$ the fractal dimension for such a configuration is $D_f = \tau_h - 1$ (see, e.g., [93]). In this sense, such a traffic jam cluster operates at the "edge of fractality." Such a hole size distribution causes logarithmic corrections to the width when $n(t)$ is given $\langle w(t) \rangle_{\text{surv}} \sim \langle n(t) \rangle_{\text{surv}} (1 + c \ln t)$.

G. STCA/1

For the STCA at $v_{\max} = 1$, from visual inspection (see Fig. 7) individual jams are not distinguishable here. Instead, the space-time plot looks much more like one from the ASEP.

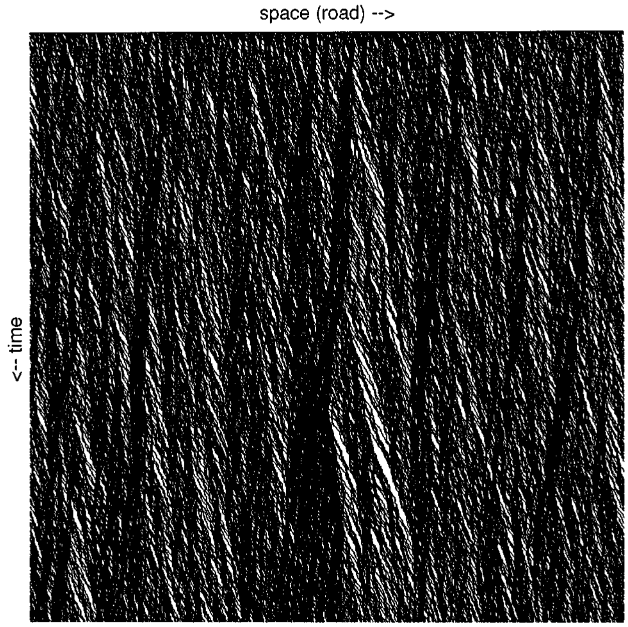

FIG. 7. Space-time plot for parallel update, $v_{\max}=1$.

The visual observation is confirmed by theoretical analysis. Schadschneider and co-workers [53–55] have performed analytical calculations for the stationary state throughput j given ρ in a closed system using n-point correlations (n-cluster method) and found that for $v_{\max}=1$ this analysis is already exact for $n=2$. This is no longer true for higher $v_{\max}$. For the ASEP, for the same analysis, the mean-field approximation, i.e., $n=1$, is already exact. The difference between the ASEP and the STCA/1 in this analysis is that in the STCA/1 one finds an effective repulsive force of range one between particles, caused by the parallel update. This helps to keep particles more equidistant than in the ASEP case, thus leading to a higher flow.

H. STCA/2

For $v_{\max}\geqslant 2$, the n-cluster analysis no longer leads to an exact solution, indicating a different dynamical regime. (In practice, though, the n-cluster analysis is already fairly close to simulation results for $n\simeq 5$.) Visual inspection of space-time plots confirms that the dynamics now is much more similar to the cruise control limit, i.e., to STCA-CC/2, than to the ASEP, except that here multiple jams exist simultaneously. Jams start spontaneously and independently of other jams because vehicles fluctuate even at maximum speed, as determined by a parameter $p_{\text{free}}\neq 0$.

The STCA displays a scaling regime near the density of maximum throughput $\rho_{j_{\max}}$. There is an upper cutoff at $t\simeq 10^4$ that was observed to depend on p_{free} [47]. One can attribute this cutoff to the nonseparation of the time scales between disturbances and the emergent traffic jams [51]. As soon as p_{free} is different from zero, the spontaneous initiation of a new jam can terminate another one. Obviously, this happens more often when p_{free} is high, which explains why the scaling region gets longer when one reduces p_{free}.

In other words, in the cruise control limit, there is a percolation-type phase transition at $\rho_c=\rho_{j_{\max}}$. Using a p_{free} larger than zero introduces an upper cutoff into this phase transition. This upper cutoff should scale with p_{free} and it should destroy the long-range connectivity of the jam clusters. This is confirmed by earlier simulations [47].

One would, though, expect that there is still a ‘‘connectivity transition’’ at higher densities, where, in spite of $p_{\text{free}}>0$, jam clusters connect to an infinite network [94]. Csányi and Kertész [45] find such a long-range connectivity of jams in the STCA/2 (using $p=1/4$) at densities much higher than $\rho_{j_{\max}}$. Further analysis is necessary to clarify the exact nature of this ‘‘connectivity transition.’’

A helpful analogy for understanding the phase separation into laminar and jammed traffic is droplet formation in a gas-liquid transition [95], where gas corresponds to the laminar phase and the droplets correspond to the jams. The gas always ‘‘tests’’ (in fluctuations) simultaneously at many positions if droplets can survive, similar to $p_{\text{free}}>0$, which tests of jams can survive. When one neglects surface tension, then droplets *cannot* survive at sub-critical density, they *can* survive at supercritical density, and they *barely* survive exactly at the critical density, making macroscopic fluctuations maximal at this point. Note that neglecting the surface tension of the droplets changes the nature of the phase transition from first order to second order.

VI. GOING BEYOND: TRAFFIC JAM DYNAMICS

All these results together put us into a position to draw a fairly consistent picture of traffic jam dynamics.

A. An intuitive starting point

Measurements of human driving behavior show that over a fairly large velocity range, g is proportional to velocity. g here is $\Delta x-L$, where Δx is the front-bumper-to-front-bumper distance (distance headway) between two cars and L is the length one car occupies (in the average) in a jam. This intuitively makes a lot of sense, since it reflects the fact that the *time* gap should be approximately the same as the delay time T, which is needed between seeing the brake lights and actually starting to brake and should therefore be largely independent of velocity (Pipe's theory; cf. [11]). (Note that traffic science traditionally does not include some ‘‘security space’’ into the definition of L [11]; therefore ‘‘gap’’ and ‘‘time gap’’ are somewhat different here.)

Field studies (cf. [11]) indeed confirm that the delay time is approximately constant for velocities between 15 and 40 miles per hour (between 24 and 64 km/h, data from the 1950s). This delay time consists of several components, including, e.g.,, reaction time or the time needed for actually pressing the brake pedal, and it is of the order of 1 sec.

Therefore, one can assume $g=Tv$. Using the average relations $1/\Delta x=\rho$ and $1/L=\rho_{\text{jam}}$ (density inside a jam), we obtain

$$v=\frac{1}{T}\left(\frac{1}{\rho}-\frac{1}{\rho_{\text{jam}}}\right). \tag{37}$$

Thus, for high density, for the current j one has

$$j_{\text{high}}=\rho v=\frac{1}{T}\left(1-\frac{\rho}{\rho_{\text{jam}}}\right). \tag{38}$$

For low density, one can assume that there is some (average) $v_{\max}$ that is independent of g for large enough spacings, and therefore, for low densities,

$$j_{\text{low}} = \rho v_{\max}. \tag{39}$$

At j_{c2} and ρ_{c2}, these two curves intersect and thereby define the maximum flow according to this model. Assuming, say, $v_{\max}=120$ km/h, $T=1.1$ sec, and $L=7.5$ m, one obtains $\rho_{c2}\approx 1/45$ m and $j_{c2}\approx 2650$ vehicles per hour per lane, which is slightly above the highest 5-min averages that are obtained in reality (e.g., [13,96]). This model is essentially equivalent to the CA-184/2 particle hopping model.

As a side remark, traffic security experts teach drivers that one should reach the position of the car ahead only after more than 2 sec, which is independent of velocity and reflects the fact that time headway is approximately equal to time gap. It is interesting to see that this would actually lead to a maximum current of two cars per second or 1800 cars per hour, much less than the up to 2400 cars per hour that are observed.

B. More realistic traffic jam dynamics

Yet, as argued further above, real traffic behaves differently from this characterization. At high densities, we do not observe the homogeneous velocity $v=g/T$ as predicted by the intuitive argument above, but relatively free flow that is interspersed by start-stop waves. This is confirmed by measurements of the j-ρ relation, where, instead of lining up on a single curve, the measurements form a fairly scattered data cloud especially in the region of the flow maximum.

For an open system, the explanation of this is as follows. Due to small fluctuations, laminar traffic at all densities will always exhibit small disturbances that can develop into jams. The inflow to the jam determines whether a jam is potentially long lived or not: Since the average outflow j_{out} is fixed by the driving dynamics, $j_{\text{in}}>j_{\text{out}}$ makes the jam (in the average) long lived, $j_{\text{in}}<j_{\text{out}}$ not. $j_{\text{in}}=j_{\text{out}}$ defines a critical point, i.e., $j_c\equiv j_{\text{out}}$ and $\rho_c\equiv\rho_{\text{out}}$, where traffic jam clusters in the average barely survive, as, e.g.,, quantified by $\langle n(t)\rangle = t^{\eta}$ with $\eta=0$.

All this is true for an open system, or a closed system that is large enough and where times are short enough so that its closed boundaries are not felt. Conversely, in a closed system, the jams ultimately absorb all the excess density $\rho_{\text{excess}}=\rho-\rho_c$. As a result, all traffic between jams operates at ρ_c. Average measurements of the long-time behavior of traffic flow can therefore show no higher flow values than $j_c=j_{\text{out}}$.

In this situation, the gas-liquid analogy (without surface tension) again is helpful. Gas can also be brought into a supercooled regime, for example, by increasing the density while keeping the temperature constant. But this state is only metastable and eventually droplets will form and, in a closed system, absorb excess density until the density surrounding the droplets is exactly at the critical point (without surface tension).

This dynamical picture explains the high variations in the short-time measurements. Measuring at a fixed position in a situation such as in Fig. 8, one can measure arbitrary combinations of supercritical laminar traffic, critical laminar traffic, jams, or traffic during acceleration or slowing down. See Fig. 9 for a comparison between short-time (300 time steps) averages and a schematic picture. Data points along the (a) branch belong to stable and laminar traffic. Data points along the (c) branch belong to still laminar, but only metastable traffic. Data points along the (d) branch belong to creeping high-density traffic.

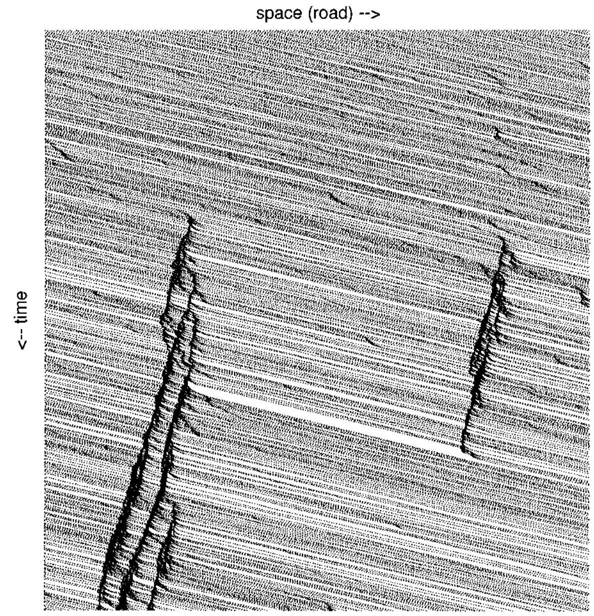

FIG. 8. Space-time plot for parallel update, $v_{\max}=5$ and $\rho=0.09$ [i.e., slightly above $\rho(j_{\max})$], starting from ordered initial conditions. The ordered state is metastable, i.e., ''survives'' for about 300 iterations until it spontaneously separates into jammed regions and into regions with $\rho=\rho(j_{\max})$.

All other data points are mixtures between regimes, where two or more regimes have been captured during the 300 iterations interval. Essentially, these data points should lie between point (b) and branch (d), yet, due to high fluctuations and due to the effects of acceleration and braking, which are not captured in the steady state arguments, we observe huge fluctuations. For example, when a car is just leaving a jam, the density decreases, but the velocity adaption is lagging somewhat behind. Therefore, the car has too low speed for the given density, leading to too low a flow value.

This scenario also makes precise the hysteresis argument of Treiterer and Myers, [97], also confirmed later [81,96]. Their measurements confirm the idea that the traffic density can go above the critical point while still being laminar, similar to the gas that can be supercooled by increasing the density. Yet, both for traffic and for supercooled gases, this state is only metastable and eventually leads to a phase separation into jams and laminar flow. Quantitative evidence of this is planned to be given in a separate paper.

Reference [81] in addition, uses catrastophe theory to interpret the measurements. Although the idea is similar in spirit to the gas-liquid analogy mentioned above, a detailed comparison does not seem possible.

The picture is also consistent with recent results both in fluid-dynamical models and mathematical car-following models for traffic flow. In Ref. [76], traffic simulations using a fluid-dynamical model starting from nearly homogeneous

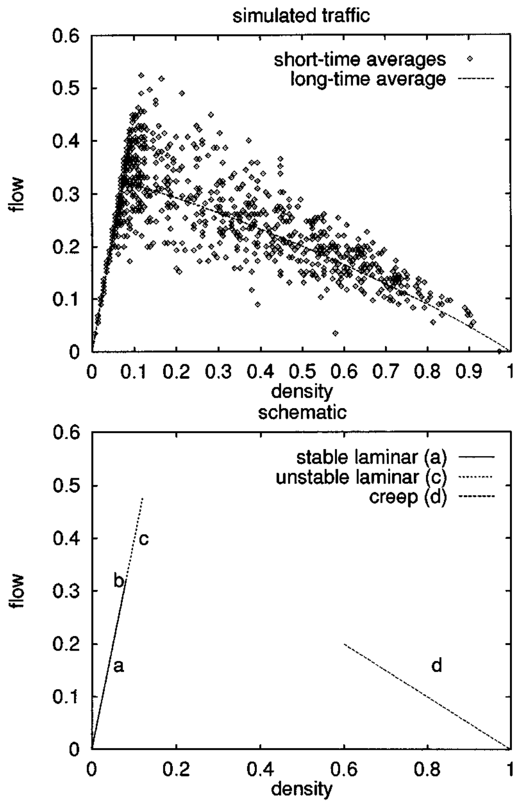

FIG. 9. Flow-density fundamental diagrams for the STCA. Top: simulation output from the STCA. Short-time averages are taken over 300 simulation steps and thus mimic the 5-min averages often taken in reality. Bottom: schematic view. (a) is the subcritical branch, (b) is the critical point, (c) is the supercritical branch, and (d) is the branch where traffic only creeps. The 5-min averages at densities between ρ_c at (b) and the creep branch are mixtures between the dynamical regimes.

conditions eventually form stable waves. Using the fluid-dynamical model one can, by usual linearization, find the parameters for the onset of instability. That is, $\langle n(t)\rangle \sim t^{\eta}$ in the particle hopping model becomes the amplitude $A(t) \sim e^{\lambda t}$ in the fluid-dynamical model and at the onset of instability $A=$ const is similar to $\langle n(t)\rangle =$const (since $\eta=0$). Therefore, the wave in the fluid-dynamical model corresponds to the *average* jam cluster in the particle hopping model.

Lee [98] explains the underlying mechanism for a model for granular media. He distinguishes ''dynamic'' from ''kinematic'' waves. Dynamic waves are found in the Kühne-Kerner-Konhäuser equations [Eqs. (22) and (23)] when the relaxation time $\tau > 0$; they are similar to sound waves in gases. Kinematic waves are found in the same equations when $\tau \rightarrow 0$, in which case the equations reduce to the Lighthill-Whitham case. The wave formation mechanism thus is that the instability first triggers the ''sound'' wave. The density inside the wave increases and outside the wave decreases until both densities are outside the unstable range. Then the kinematic mode takes over.

Kurtze and Hong [80] make this more precise for the Kühne-Kerner-Konhäuser equations: Below the critical density, the kinematic wave with wave velocity $c=j'=dj/d\rho$ is the only solution of the linearized equation and this solution is stable. At the critical density, this solution bifurcates into two unstable solutions with wave velocity $c=j' \pm \epsilon$, where $\epsilon \rightarrow 0$ for $\rho \searrow \rho_c$ and ϵ is the equivalent of the speed of sound.

Actually, things might be somewhat more complicated. Kerner and Konhäuser also find a large amplitude instability, which exists already at lower densities than the instability obtained from the linearized equations [77]. That opens the discussion of which of these two instabilities corresponds to the instability of the STCA or if such a detailed comparison is possible at all. Intuitively, one would assume that the large-amplitude instability is the relevant instability for a noisy system such as the STCA. Bando and co-workers [26] also find the separation of traffic into laminar and jammed phases in a deterministic continuous mathematical car-following model.

VII. SUMMARY AND CONSEQUENCES FOR TRAFFIC SIMULATION MODELS

These findings have some fairly far-reaching implications for traffic simulation models.

(i) Robust numerics. Particle hopping models, which seem at the first glance as too rough an approximation of reality, include the same range of dynamic phenomena as the most advanced fluid-dynamical models for traffic flow to date. Yet, particle hopping models offer some distinctive advantages for practical simulations. Particle hopping models are known to be numerically robust especially in complex geometries and realistic road networks with all their interconnections, etc., certainly are complex geometries. Practical road network implementations of the fluid-dynamical theory are so far only using the Lighthill-Whitham equations, which are (without diffusion) marginally stable and can certainly be made stable by using a stable numerical discretization scheme.

(ii) ''Universality.'' Intuitively, a relatively simple microscopic model should be able to show the essential features of traffic jams. One might even speculate that the critical exponents of traffic jam formation are *universal*, i.e., robust against changes in microscopic rules. This speculation is backed up by the fact that the exponents of our model can be theoretically explained. The consequence for traffic simulation is that, as long as one expects certain simple aspects of traffic jam formation to be realistic enough for the problem under consideration, e.g.,, for large-scale questions, *the simplest possible model will be sufficient for the task*, thus saving human and computational resources.

(iii) Towards minimal models. The present results show that closeup car-following behavior is not the most important aspect of traffic to model. The important crucial aspect is to model *deviations* from the optimal (smooth) behavior and the ways in which they lead to jam formation. Another important aspect, which seems far from obvious, is the *acceleration behavior*, especially when there are other cars ahead, since it is the acceleration behavior that mostly determines the maximum flow out of a jam (which may be a simple traffic light). Therefore, investigations such as this paper *are*

important for microscopic modeling as long as one does not have the perfect model of driving or the computational resources to run it.

(iv) Traffic dynamics. Fast running and easy to implement particle hopping models can be very useful in interpreting measurements. Measurements such as for the traditional 5-min-averaged fundamental diagrams (flow vs density vs velocity) have increasingly recognized the fact that the dynamics around the measurement site has an extreme influence on the outcome of the measurements, thus making the results far from universal. This point is planned to be further discussed in another paper.

(v) Microscopic simulations. Particle hopping models are inherently microscopic, which allows one to add individual properties to each car such as the identity of travelers, route plan, and engine temperature (for emission modeling). These properties are imperative for the kind of traffic models that are needed in current policy evaluation processes.

(vi) Stochasticity and fluctuations. Last but not least, particle hopping models are stochastic in nature, thus producing different results when using different random seeds even when starting from identical initial conditions. At first, this is certainly considered a disadvantage from the point of view of policy makers or traffic engineers. However, the traffic system is inherently stochastic and the variance of the outcomes is an important variable itself. How will we be able to distinguish reliable from unreliable predictions without knowing something about the range of possible outcomes? Furthermore, there is reason to believe that the average over several stochastic runs will *not* be identical to a deterministic run. Imagine, for example, a case where in a deterministic model, a queue at one intersection has a backspill that, on the average, *just* does not reach another intersection. (By queue I mean a queue with spatial extension. This is different from the use of the word in queuing theory.) In the stochastic model, the maximum length of this queue will, between different simulation runs, fluctuate around its average value, thus backspilling into the other intersection in nearly 50% of all runs. Since this possibly disrupts traffic in this other intersection, this can cause long-range effects and network breakdown.

VIII. SOME OPEN QUESTIONS

Many open questions remain, though. The following are examples.

What is the exact relation between average cluster growth in CA models, wave amplitude growth in fluid-dynamical models, droplet growth in the liquid-gas transition interpretation, and phase space portraits in car following models?

Is there a hydrodynamical limit for the STCA? If so, how can it be proven to be correct? Do critical exponents help here?

What is the exact relation to granular media? Pöschel has both observed and simulated similar waves for sand falling down in a narrow tube [99]. He has also found in the simulations the bistability leading to laminar flow or to jam waves depending on the initial conditions. Peng and Herrmann found similar waves in lattice gas automata simulations of the same situation [100]. Lee and Leibig [101,98] have related these waves to a fluid-dynamical theory that is similar to the Kühne-Kerner-Konhäuser theory for traffic flow. Schäfer finds a similar phase transition as the one stressed in this paper for simulated granular flow, except that above the critical point, the flow is exactly zero [102]; supposedly, such a flow-density relation would also support the same overall dynamics.

What is the *minimal* ingredient for the instability that causes the traffic breakdown? Both the car-following models (CA and continuous) and the fluid-dynamical approach have produced the instability after adding an inertia term. Yet, Goldhirsch and Zanetti point out that an inverse temperature effect is responsible for the clustering [82].

What is the exact relation of particle hopping to car-following models, either continuous in space only or continuous in both space and time [38]? The range of phenomena that are captured seems comparable. Yet, on one hand, most car-following models investigated so far do not include randomness; on the other hand, it is unclear what the better resolution actually buys in terms of additional insight. Komatsu and Sasa [103] derive a fluid-dynamical equation from a car-following model.

Can one say more about universality than in Sec. VII?

What can one-dimensional theory say about two-dimensional problems, such as they are regularly encountered for urban traffic problems? A series of papers (see, e.g., [64–66]) have used cellular automata techniques for building models for town traffic. These models use the CA-184/1 model for driving dynamics, but add elements for directional changes. Molera and co-workers have built a theory for their two-dimensional model [104] and their flow equation is essentially a two-dimensional version of the Lighthill-Whitham equation *with a quadratic flow-density relation.* That means that adding stochastic directional changes would change the model from CA-184 type to the ASEP type.

What is the relation to $1/f$ noise? Musha and Higuchi have measured $1/f$ noise in the power spectrum of a car detector time series [23]. They explained this by a noisy Burgers equation, in a way, though, that differs from Krug's interpretation [90]. Nagel and Paczuski [51] have predicted a precise $1/f$ law for the power spectrum of the density time series, which was roughly confirmed by simulations for STCA-CC/2. Yet, Nagel and Herrmann find, using a continuous car-following model and following the traffic movement, a $1/f^{\alpha}$ law, with $\alpha \approx 1.3$ [105]. Car following is slightly different from the particle hopping models in this paper; but if the arguments in [51] were entirely correct, this should not matter. Choi and Lee find $1/f$-like behavior in simulations of slightly modified versions of the fluid-dynamical equations for traffic [106]; Zhang and Hu find $1/f$-like behavior in simulations of a discrete-time–continuous-space model [107]. Understanding $1/f$ noise behavior would be helpful because it would be much easier to measure in reality than, say, lifetime distributions [47,51].

What is the meaning to the ongoing discussion about the value of synchronous updating for explaining physical phenomena? Huberman and Glance [108] have reissued the warning that parallel updating may produce artifacts and that usually stochastic asynchronous updating would be a better approximation of reality. Yet, for the traffic case, it is clear from this paper that the (synchronous) STCA produces a much better model for reality than the (asynchronous) ASEP. One would probably have to go to much higher spatial and temporal resolutions (and thus lose all the computational advantages) when one wanted to build a stochastically updating model of traffic.

This paper treats single-lane models only. It is though interesting to note that the empirical evidence that backs up particle hopping models for traffic [35] stems from multilane highways. I expect, also in agreement with our multilane simulation results [57–62], that *homogeneous* multi-lane traffic (symmetric lane-changing rules, all vehicles the same) behaves similarly to the single-lane models presented here. Deviations from homogeneity will introduce more and more additional effects, which will have to be investigated in detail.

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